A Reminder on Home Test 1

Please be reminded that Home Test 1 will be held **next Thursday (29/10/2020)**. There will **still** have tutorial and lecture in the morning. Observe the following details of the test:

- Duration: 24 hours (29/10 12:00 noon to 30/10 12:00 noon)
- Content: Section 1 - 5 of Lecture Note. (Reference: Chapter 2 and 3 of textbook)
- Delivery: The test paper will be sent to the **university email account** at 12:00 noon.
- Submission: Submit **one PDF file** to **Blackboard**. (Same as homework assignments)

A review exercise will be posted later. The suggest solution will be post on 26/10/2020 (Monday). I will also explain it in next week's tutorial.

Consequences of the Completeness Property

Recall the **Completeness Property** of $\mathbb{R}$, which appears in the beginning of the course:

The Completeness Property of $\mathbb{R}$ (c.f. 2.3.6)**.** *Every bounded above non-empty subset of* $\mathbb{R}$ *has a supremum in* $\mathbb{R}$*.*

We have learnt the following theorems, which can be all derived from the above axiom:

Monotone Convergence Theorem (c.f. 3.3.2)**.** *A monotone sequence* (x_n) *of real numbers is convergent if and only if it is bounded. Moreover,*

- *if* (x_n) *is increasing, then* $\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$.
- *if* (x_n) *is decreasing, then* $\lim(x_n) = \inf\{x_n : n \in \mathbb{N}\}$.

Nested Intervals Property (c.f. 2.5.2)**.** *Suppose* $I_n = [a_n, b_n]$ *for each* $n \in \mathbb{N}$ *is a nested sequence of closed bounded intervals. i.e.,*

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots .$$

Then there exists a number $\xi \in \mathbb{R}$ *such that* $\xi \in I_n$ *for all* $n \in \mathbb{N}$*. i.e.,* $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$*. Furthermore, if* $\lim(b_n - a_n) = 0$*, then the number* ξ *is unique.*

Remark. The condition that the intervals being **closed** and **bounded** cannot be dropped.

The Bolzano-Weierstrass Theorem. *A bounded sequence of real numbers has a convergent subsequence.*

The proofs of the main theorems are important. You are encouraged to revise and understand them. Other than the above theorems, there is one more consequence to the completeness property of $\mathbb{R}$. It will be covered in the next section.

Cauchy Sequences

Definition (c.f. Definition 3.5.1)**.** A sequence of real numbers (x_n) is said to be a *Cauchy sequence* if for every $\varepsilon > 0$, there exist a natural number $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon, \qquad \forall n, m \geq N.$$

Remark. Since $|x_n - x_m| = |x_m - x_n|$, we can simply check for the case $n \geq m \geq N$.

Example 1 (c.f. Example 3.5.2)**.** $(1/n)$ is a Cauchy sequence.

Proof. We need to show that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon, \qquad \forall n, m \geq N.$$

Note that if $n, m \geq N$,

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|-\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}.$$

Let $\varepsilon > 0$. By **Archimedean Property**, there exists $N \in \mathbb{N}$ such that $1/N < \varepsilon/2$. Then

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{2}{N} < \varepsilon, \quad \forall n, m \geq N.$$

This shows that $(1/n)$ is a Cauchy sequence by definition. □

Example 2. $(1 - (-1)^n)$ is not a Cauchy sequence.

Proof. We need to show that there exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$, there exists $n, m \geq N$ such that

$$|(1 - (-1)^n) - (1 - (-1)^m)| \geq \varepsilon.$$

Notice that the sequence is alternating between 0 and 2, so any successive terms are differed by 2. Take $\varepsilon = 2 > 0$. Then for any $N \in \mathbb{N}$, take $n = N + 1$ and $m = N$. Then

$$|(1 - (-1)^n) - (1 - (-1)^m)| = 2 \geq \varepsilon.$$

It follows that $(1 - (-1)^n)$ is not a Cauchy sequence by definition. □

With the notation of a Cauchy sequence, we can now state:

Cauchy Convergence Criterion (c.f. 3.5.5)**.** *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Remark. A sequence being Cauchy is a formally weaker condition than being convergent. It is easier for us to check whether or not a sequence is Cauchy than checking its convergence because we don't need to specify the limit.

Example 3 (c.f. Section 3.5, Ex.5)**.** Let $x_n = \sqrt{n}$. Show that $\lim |x_{n+1} - x_n| = 0$, but (x_n) is not a Cauchy sequence.

Remark. This tells us that we cannot simply check the difference between successive terms.

Solution. Direct calculation gives:

$$\begin{aligned}\lim_{n\to\infty} |x_{n+1} - x_n| &= \lim_{n\to\infty} |\sqrt{n+1} - \sqrt{n}| \\ &= \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \lim_{n\to\infty} \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\ &= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ &= 0\end{aligned}$$

To see that (x_n) is not a Cauchy sequence, simply note that this sequence is unbounded and the result follows. To see this from definition, take $\varepsilon = 1 > 0$. For any $N \in \mathbb{N}$, take $n = (N+1)^2$ and $m = N^2$, then

$$|x_n - x_m| = |\sqrt{(N+1)^2} - \sqrt{N^2}| = |N + 1 - N| = 1 \geq \varepsilon.$$

Exercise. Show that $(\sqrt{n})$ is an unbounded sequence.

Example 4 (c.f. Section 3.5, Ex.9)**.** Suppose $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$. Show that (x_n) is a Cauchy sequence.

Solution. Notice that whenever $n \geq m \geq N$,

$$\begin{aligned}|x_n - x_m| &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\ &< r^{n-1} + r^{n-2} + \cdots + r^m \\ &< r^m \cdot (1 + r + r^2 + \cdots) \\ &\leq \frac{r^N}{1-r}\end{aligned}$$

Let $\varepsilon > 0$. Since $0 < r < 1$, (r^n) converges to 0, there exists $N \in \mathbb{N}$ such that

$$r^N < (1-r)\varepsilon \iff \frac{r^N}{1-r} < \varepsilon.$$

Hence whenever $n \geq m \geq N$, we have $|x_n - x_m| < \varepsilon$. The result follows.

Exercise. Prove that if $0 < r < 1$, then $\lim r^n = 0$. (Hint: Let $s = (1/r) - 1 > 0$. See Example 6, Tutorial 2.)

Example 5 (c.f. Section 3.5, Ex.10)**.** If $x_1 < x_2$ are arbitrary real numbers and

$$x_n = \frac{1}{2}(x_{n-2} + x_{n-1}) \qquad \text{for } n > 2,$$

show that (x_n) is convergent. What is its limit?

Solution. To show that the sequence is convergent, it suffice to show that it is a Cauchy sequence by **Cauchy Convergence Criterion**. Since the terms are constructed by averaging, for each $n \in \mathbb{N}$, we have

$$|x_{n+1} - x_n| = \frac{|x_n - x_{n-1}|}{2} = \cdots = \frac{|x_3 - x_2|}{2^{n-2}} = \frac{x_2 - x_1}{2^{n-1}}.$$

By a similar argument as the above example, we can show that (x_n) is Cauchy and hence convergent. To find the limit of this sequence, note that we cannot find the limit by solving

$$x = \frac{1}{2}(x + x).$$

Instead, since the sequence is convergent, all of its subsequence will converge to the same limit. In particular, we consider the subsequence (x_{2n+1}). Observe that for all $n \in \mathbb{N}$,

$$x_{2n+1} = x_1 + \frac{x_2 - x_1}{2}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^{n-1}}\right) = x_1 + \frac{x_2 - x_1}{2}\sum_{k=0}^{n-1}\frac{1}{4^k}. \tag{1}$$

Hence we can calculate the limit of (x_n) by

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n+1} = x_1 + \frac{x_2 - x_1}{2}\sum_{k=0}^{\infty}\frac{1}{4^k} = \frac{1}{3}x_1 + \frac{2}{3}x_2.$$

Exercise. Prove (1) by induction.